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The Unit Group In Certain Hecke Algebras

[1]Yannizhou, [2]xigou Zhang, [3]Zhaojin Zhou
[1, 2, 3] College of Mathematics and Information Science, JXNU Nanchang,P.R. China 330022

Abstract: In this paper, we introduce some properties of unit group in the degenerate Hecke algebra.

Keywords: Degenerate Hecke algebras; invertible elements; unit group; Coxeter groups.

Introduction

A Coxeter System is a pair (W, S) consisting of a group W and a set of generators S, subject only to relations of the form $(ss')^{m(s,s')} = 1$, where $m(s,s') = 1$, $m(s,s') = m(s',s) \geq 2$ for $s \neq s'$ in S.

An arbitray $w \in W$ can be written as a product of elements in S, say $w = s_1 \cdots s_r$ (where $s_i \in S$). Define the length $l(w)$ of w to be the smallest r for which such an expression exists, and call the expression reduced.

We begin with a very general constrution of associative algebras over a commutative ring A (with1). Such an algebra will have a free $A-$basis parametrized by the element of W, together with a multiplication law which reflects in a certain way the multiplication in W. The algebra will also depend on some parameters $a_s, b_s \in A$ $(s \in S)$, subject only to the requirement that $a_s = b_t$ and $b_s = b_t$ whenever s and t are conjugate in W, The starting point for the constrution is a free $A-$module ε on the set W, with basis elements denoted T_w $(w \in W)$.

THEOREM 1.1(see [1]) Given elements a_s, b_s as above, there exists a unique structure of associative $A-$algebra on the free $A-$module ε, with T_1 acting as the identity, such that the identity, such that the following conditions hold for all $s \in S, w \in W$:

$$T_s T_w = T_{sw} \quad \text{if } l(sw) > l(w), \qquad (1)$$

$$T_s T_w = a_s T_w + b_s T_{sw} \quad \text{if } l(sw) < l(w). \qquad (2)$$

The algebra described by the theorem, denoted $\varepsilon_A(a_s, b_s)$, is called a generic algebra. Having constructed generic algebras $\varepsilon_A(a_s, b_s)$ over an arbitrary commutative ring A. Now let A be the ring $Z[q, q^{-1}]$ of Laurent polynonianls over Z in the indeterminate q. With the further convention that $a_s = q - 1$ and $b_s = q$ for all $s \in S$, we write H for the resulting generic agebra and call it the Hecker algebra of W. In paticular, we call it the degenerate Hecker algebra of W if $q = 0$.The relations (1) and (2) become:

$$T_s T_w = T_{sw} \quad \text{if } l(sw) > l(w),$$

$$T_s T_w = -T_w \quad \text{if } l(sw) < l(w).$$

For the right-handed version, we have the relations as follows

$$T_w T_s = T_{ws} \quad \text{if } l(ws) > l(w),$$

$$T_w T_s = -T_w \quad \text{if } l(ws) < l(w).$$

Main Result and Their Proofs

PROPOSITION 2.1 Let $w_1 \in W$,If there exists $w_2 \in W$ such that $T_{w_1} T_{w_2} = T_1$, then $w_1 = w_2 = 1$.

Proof. Suppose that $T_{w_1} T_{w_2} = \pm T_w$ it is clear that $l(w) \geq l(w_1)$ and $l(w) \geq l(w_2)$. Moreover, by the hypothesis $T_{w_1} T_{w_2} = T_1$, so we can get $w_1 = w_2 = 1$.

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Given W , with $|W| = n < \infty$, say $W = \{w_1, w_2, \cdots, w_n\}$, where $l(w_1) \le l(w_2) \le \cdots \le l(w_n)$ and w_1 is the identity element.

PROPOSITION 2.2 Let $h = a_1T_{w_1} + a_2T_{w_2} + \cdots + a_nT_{w_n}$, $h_1 = b_1T_{w_1} + b_2T_{w_2} + \cdots + b_nT_{w_n}$ $(l(w_1) \le l(w_2) \le \cdots \le l(w_n))$, if $hh_1 = T_1$, then $w_1 = 1$, $a_1b_1 = 1$ and $T_{w_m}T_{w_n} = \pm T_{w_j}$, $T_{w_n}T_{w_m} = \pm T_{w_j}$, where $m \le n \le j$ and $j \ne 1$, all coefficients of these products in the sum is equal to 0.

Proof. According to $hh_1 = T_1$, it is clear to get these results.

Given $h = \sum_{w \in W} a_wT_w = a_1T_{w_1} + a_2T_{w_2} + \cdots + a_nT_{w_n}$, $h_1 = \sum_{w \in W} b_wT_w = b_1T_{w_1} + \cdots + b_nT_{w_n}$,where $a_{w_i} = a_i$, $b_{w_i} = b_i$, we denote $(a_1, a_2, \cdots a_n)^T$ by B . Suppose that $hh_1 = T_1$, we will describe a method to compute $b_1, \cdots, b_n$.

For every $w_i \in W$ $(1 \le i \le n)$, we define by T_{w_i} a linear transformation of H and $T_{w_i} : T_{w_i}(T_w) = T_{w_i}T_w$ $(w \in W)$, let M_i be a matrix of a linear transformation T_{w_i} under the basis $\{T_{w_1}, T_{w_2}, \cdots T_{w_n}\}$, which show that M_i is a lower triangular matrix.

Remark: $W = \{w_1, w_2, \cdots, w_n\}$, for all $h \in \sum_{w \in W} a_wT_{w_i}$, we write $h = a_1T_{w_1} + a_2T_{w_2} + \cdots + a_nT_{w_n}$, where $a_i \in Z$. h can be written in n ! different way, but computing the final result of h_1 can not change, to simplify calculation, we seek a partial ordering $l(w_1) \le l(w_2) \le \cdots \le l(w_n)$ in W . some results are described in the below.

$$T_{w_i}(T_{w_1}, T_{w_2}, \cdots, T_{w_n}) = (T_{w_1}, T_{w_2}, \cdots, T_{w_n})M_i \qquad (3)$$

Let a_i act on (3), then

$$a_iT_{w_i}(T_{w_1}, T_{w_2}, \cdots, T_{w_n}) = (T_{w_1}, T_{w_2}, \cdots, T_{w_n})a_iM_i$$

We can conclude

$$(a_1T_{w_1} + a_2T_{w_2} + \cdots + a_nT_{w_n})(T_{w_1}, T_{w_2}, \cdots T_{w_n}) = (T_{w_1}, T_{w_2}, \cdots T_{w_n})(a_1M_1 + \cdots + a_nM_n)$$

$$(T_{w_1}, T_{w_2}, \cdots T_{w_n})\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}(T_{w_1}, T_{w_2}, \cdots T_{w_n}) = (T_{w_1}, T_{w_2}, \cdots T_{w_n})(a_1M_1 + \cdots + a_nM_n)$$

$$hh_1 = (T_{w_1}, T_{w_2}, \cdots T_{w_n})\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}(T_{w_1}, T_{w_2}, \cdots T_{w_n})\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (T_{w_1}, T_{w_2}, \cdots T_{w_n})(a_1M_1 + \cdots + a_nM_n)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Suppose that $hh_1 = T_1$, the following condition is satisfied:

$$(T_{w_1}, T_{w_2}, \cdots T_{w_n})(a_1M_1 + \cdots + a_nM_n)\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = (T_{w_1}, T_{w_2}, \cdots T_{w_n})\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Let $h = \sum a_wT_w \in H$, there exists some $h_1 \in \sum b_wT_w \in H$ such that $hh_1 = T_1$. it implies that

$$(a_1M_1 + \cdots + a_nM_n)X = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Write $V_1 = (1,0,\cdots,0)^T$, then $(a_1M_1 + \cdots + a_nM_n)X = V_1$. In particular, let $h = h_1$, $h_1^2 = T_1$, which satisfy $(a_1M_1 + \cdots + a_nM_n)B = V_1$.

Example

3.1. $W = A_1$, $T_{w_1} = T_1, T_{w_2} = T_{s_1}$ .Obviously, $T_1T_1 = T_1, T_1T_{s_1} = T_{s_1}, T_{s_1}T_1 = T_{s_1}, T_{s_1}T_{s_1} = -T_{s_1}$. Thus

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$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$$

Let $h = a_1 T_{w_1} + a_2 T_{w_2}$, $h = b_1 T_{w_1} + b_2 T_{w_2}$, then

$$(a_1 M_1 + a_2 M_2)\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Or, equivalently

$$\begin{cases} a_1 b_1 = 1 \\ a_2 b_1 + (a_1 - a_2) b_2 = 0 \end{cases}$$

We can obtain

$a_1 = 1, b_1 = 1, a_2 = 0, b_2 = 0$; or $a_1 = -1, b_1 = -1, a_2 = 0, b_2 = 0$; or $a_1 = 1, b_1 = 1$, $a_2 = 2, b_2 = 2$;

or $a_1 = -1, b_1 = -1, a_2 = -2, b_2 = -2$;

Consider the case that $h = h_1$, we have

$$\begin{cases} a_1^2 = 1 \\ a_2 a_1 + (a_1 - a_2) a_2 = 0 \end{cases}$$

We can get

$$\begin{cases} a_1 = 1 \\ a_2 = 0 \end{cases} \begin{cases} a_1 = -1 \\ a_2 = 0 \end{cases} \begin{cases} a_1 = 1 \\ a_2 = 2 \end{cases} \begin{cases} a_1 = -1 \\ a_2 = -2 \end{cases}$$

Therefore

$T_1^2 = T_1$, $(-T_1)^2 = T_1$, $(T_1 + 2T_{s_1})^2 = T_1$, $(-T_1 - 2T_{s_1})^2 = T_1$.

Given W as above, let U be the unity group which consists of all invertible elements in H, we will know that $T_1 \in U$, $-T_1 \in U$ and some $h = \sum_{i=1}^{n} a_i T_{w_i}$ may lie in U, assume that $h \in U$, it not only satisfies $hh_1 = T_1$, but also $h_1 h = T_1$, where $h_1 = \sum_{i=1}^{n} b_i T_{w_i}$, since T_{w_i} and T_{w_j} $(i \neq j)$ may not commute. if $h \in U$ iff $hh_1 = h_1 h = T_1$, in particular, if $h^2 = T_1$, then $h \in U$. When $W = A_1$, for any $h \in \sum a_w T_w, h_1 \in \sum a_w T_w$, h and h_1 can commute, since T_{w_i} and T_{w_j} $(i \neq j)$ can commute. From the previous example, we can obtain that $(T_1 + 2T_{s_1}) \in U$ and $(-T_1 - 2T_{s_1}) \in U$. Consider an example in the below, where not all T_{w_i} and T_{w_j} $(i \neq j)$ can commute.

3.2. let $W = A_2$, $T_{w_1} = T_1, T_{w_2} = T_{s_1}, T_{w_3} = T_{s_2}, T_{w_4} = T_{s_1 s_2}, T_{w_5} = T_{s_2 s_1}, T_{w_6} = T_{s_1 s_2 s_1}$.

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 \end{bmatrix}, M_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}, M_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}.$$

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Let $h=\sum_{i=1}^{6}a_iT_{w_i}$, suppose that $h\in U$, then there exists $h_1=\sum_{i=1}^{6}b_iT_{w_i}$ such that $hh_1=h_1h=T_1$, which satisfy

$$(a_1M_1+\cdots+a_6M_6)\begin{pmatrix}b_1\\b_2\\\vdots\\b_6\end{pmatrix}=V_1 \text{ and } (b_1M_1+\cdots+b_6M_6)\begin{pmatrix}a_1\\a_2\\\vdots\\a_6\end{pmatrix}=V_1$$

Thus , we can obtain

$a_1=1,b_1=1,a_2=2,b_2=2,a_3=2,b_3=2,a_4+b_4=4,a_5+b_5=4,$

$16-3(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0;$

Or

$a_1=1,b_1=1,a_2=2,b_2=2,a_3=0,b_3=0,a_4=b_4,a_5=b_5,$

$2(a_4+b_5)-(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0;$

Or

$a_1=1,b_1=1,a_2=0,b_2=0,a_3=2,b_3=2,a_4=b_4,a_5=b_5,$

$2(a_5+b_4)-(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0;$

Or $a_1=1,b_1=1,a_2=0,b_2=0,a_3=0,b_3=0,a_4+b_4=0,a_5+b_5=0,$ $a_6+b_6+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0$;

Or $a_1=-1,b_1=-1,a_2=-2,b_2=-2,a_3=-2,b_3=-2,a_4+b_4=-4,a_5+b_5=-4,$

$16+3(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0;$

Or

$a_1=-1,b_1=-1,a_2=-2,b_2=-2,a_3=0,b_3=0,a_4=b_4,a_5=b_5,$

$-2(a_4+b_5)+(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0;$

Or

$a_1=-1,b_1=-1,a_2=0,b_2=0,a_3=-2,b_3=-2,a_4=b_4,a_5=b_5,$

$-2(a_5+b_4)+(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0;$

Or

$a_1=-1,b_1=-1,a_2=0,b_2=0,a_3=0,b_3=0,a_4+b_4=0,a_5+b_5=0,$ $-(a_6+b_6)+(-a_4-a_5+a_6)(b_4+b_5-b_6)=0$;

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