International Journal of Latest Research In Engineering and Computing (IJLREC) ISSN:2347-6540
Volume 1, Issue 2 : Page No15-22, November-December 2013
www.ijlrec.com

Positive Solutions To Nonlinear Semipositone Boundary Value Problems*

Shixia Luan, Hua Su
School of Mathematical Sciences, Qufu Normal University, Qufu Shandong, 273165, China
School of Mathematics and Quantitative Economics, Shandong University of Finance and Economics, Jinan Shandong 250014, China

Abstract

In this paper, we investigate the following third-order three-point semipositone boundary value problems:

$$\begin{cases} u'''(t)-f(t,u)=0, \quad t\in(0,1); \\ u(0)=u'(\eta)=u''(1)=0, \end{cases}$$

Under the conditions that the nonlinear term $f(t,u):(0,1)\times(0,+\infty)\to(-\infty,+\infty)$ is continuous, i.e., we allow that the nonlinear term f is both semipositone and lower unbounded. By using the fixed-point index theory, the existence of positive solution and many positive solutions are obtained.

Key words : Third-order three-point semipositone boundary value problems, lower unbounded, positive solutions,fixed-point index theory.

2000 MR. Subiect Classification 34B 16.

Introduction

In this paper, we study the following third-order three-point semipositone boundary value problems (SBVP):

$$\begin{cases} u'''(t)-f(t,u)=0, \ t\in(0,1); \\ u(0)=u'(\eta)=u''(1)=0, \end{cases} \qquad (1.1)$$

Where $1/2<\eta<1$, $f(t,u):(0,1)\times[0,+\infty)\to(-\infty,+\infty)$.

In recent years, the existence of positive solutions for nonlinear boundary value problems received wide attention. But they all request the positive continuous or lower bounded of the nonlinear term (see [1-7, 9]). For example, in 1998, D.Aanderson [6] considered the following problem (1.2) and obtained an existence result about positive solutions when $f(t,l)=g(l)$ and $g:[0,+\infty)\to[0,+\infty)$.Recently, Yao [7] has investigated (1.1) when f is semipositone and lower bounded, and he obtained the existence theorem.

In this paper, by constructing a new Lemma which is important to resolve the semipositone boundary value problem SBVP (1.1) (See Lemma 2.4 and the proof in section 3, 4), we research the existence of positive solutions for semipositone boundary value problem SBVP (1.1) under the conditions that the nonlinear term $f(t,u):[0,1]\times(0,+\infty)\to(-\infty,+\infty)$ is continuous, i.e., we allow that the nonlinear term f is both semipositone and lower unbounded.

Our main tool of this paper is the following fixed point index theory.

Theorem $1.1^{[8]}$.Suppose E is a real Banach space, $K \subset E$ is a cone, let $\Omega r = \{u \in K : \|u\| \le r\}$. Let operator $T : \Omega r \to K$ be completely continuous and satisfy $Tx \ne x$, $\forall x \in \Omega r$. Then

（i）If $\|Tx\| \le \|x\|$, $\forall x \in \Omega r$, then $i(T, \Omega r, K) = 1$;

（ii）If $\|Tx\| \ge \|x\|, \forall x \in \Omega r$,then $i(T, \Omega r, K) = 0$;

This paper is organized as follows. In section 2, we present some preliminaries and lemmas that will be used to prove our main results. In section 3, we discuss the existence of single solution of the SBVP (1.1). In section 4, we study the existence of at least two solutions of the SBVP (1.1). In section 5, we give an example as an application.

2 Preliminaries and Lemmas

Let $I = [0,1], E = C[I, R]$, then E is a Banach space with $\|x\| = \max\limits_{t \in I} |x(t)|$. We also introduce the space $L^1(0,1)$ with norm $\|x\|_1 = \int_0^1 |x(t)| dt$.

Throughout this paper, we shall use the following notation:

$$G(t,s) = \begin{cases} ts - \frac{1}{2}t^2, & 0 \le s \le \eta,\ 0 \le t \le s; \\ \frac{1}{2}s^2, & 0 \le s \le \eta,\ 0 \le s \le t; \\ \eta t - \frac{1}{2}t^2, & \eta \le s \le 1,\ 0 \le t \le s; \\ \frac{1}{2}s^2 - ts + \eta t, & \eta \le s \le 1,\ 0 \le s \le t. \end{cases}$$

It is well known that $G(t,s)$ is the Green's function of homogeneous boundary value problem:

$$\begin{cases} u'''(t) = 0,\ 0 \le t \le 1; \\ u(0) = u'(\eta) = u''(1) = 0. \end{cases}$$

Obviously, $G(t,s)$ is nonnegative continuous function.

By direct account, we can easily obtain the following results.

Lemma 2.1 ([7]). $G(t,s)$ defined as above have the following properties:

$$q(t)J(s) \le G(t,s) \le J(s),\ 0 \le t,\ s \le 1,$$

where

$$J(s) = \max_{t \in I} G(t,s) = \begin{cases} \frac{1}{2}s^2, & 0 \le s \le \eta, \\ \frac{1}{2}\eta^2, & \eta \le s \le 1, \end{cases} \qquad q(t) = \begin{cases} \eta t, & 0 \le t \le \eta; \\ 2\eta t - t^2, & \eta \le t \le 1, \end{cases}$$

Lemma 2.2. For the unique position solution $u(t)$ of the following BVP:

$$\begin{cases} u'''(t) = h(t), & 0 < t < 1, \\ u(0) = u'(\eta) = u''(1) = 0, \end{cases}$$

Where $h \in L^1(0,1)$, $h \ge 0$. Then

$$u(t) \ge \|u\| q(t),\ 0 \le t \le 1.$$

Proof. By $q(t)J(s) \le G(t,s) \le J(s),\ 0 \le t,\ s \le 1,$ we have

$$u(t) = \int_0^1 G(t,s)h(s)ds \le \int_0^1 J(s)h(s)ds,$$

so

$$\|u\| \le \int_0^1 J(s)h(s)ds.$$

Therefore, for $0 \le t \le 1$, we have

$$u(t) = \int_0^1 G(t,s)h(s)ds \ge q(t)\int_0^1 J(s))h(s)ds \ge \|u\| q(t).$$

This completes the proof of Lemma 2.2.

Lemma 2.3 For the unique position solution $u(t)$ of the following BVP: $\begin{cases} u'''(t)=h(t), & 0<t<1, \\ u(0)=u'(\eta)=u''(1)=0, \end{cases}$

where $h \in L^1(0,1)$, $h \geq 0$. Then, for any $\theta \in (0,1/2)$,there exists constant $\sigma > 0$ such that

$$u(t) \geq \sigma \|u\|, \ \theta \leq t \leq 1-\theta.$$

Proof. Let $\sigma = \max\limits_{\theta \leq t \leq 1-\theta} q(t)$ and then by the Lemma 202, we can obtain the results. This completes the proof of Lemma 2.3.

Lemma 2.4. Suppose that $\overline{w}(t)$ is the solution of the following BVP,

$$\begin{cases} u'''(t)=M(t), & t\in(0,1)\ ; \\ u(0)=u'(\eta)=u''(1)=0, \end{cases}$$

Where $M(t) \in L^1(0,1)$, $h \geq 0$. Then, there constant $C \geq 1$ such that

$$\overline{w}(t) \leq C\|M\|_1 q(t), \ 0 \leq t \leq 1.$$

Proof. For $t \in [\eta, 1]$, we can have

$$\begin{aligned} \overline{w}(t) &= \int_0^1 G(t,s)M(s)ds \\ &= \int_0^\eta \frac{1}{2}s^2 M(s)ds + \int_\eta^t (\frac{1}{2}s^2 - ts + \eta t)M(s)ds + \int_t^1 (\eta t - \frac{1}{2}t^2)M(s)ds \\ &\leq \int_0^t \frac{1}{2}s^2 M(s)ds + \int_\eta^t (\frac{1}{2}s^2 - ts + \eta t)M(s)ds + \int_0^1 (\eta t - \frac{1}{2}t^2)M(s)ds \\ &\leq \left[\frac{1}{2}t^2 + (\frac{1}{2}t^2 - t\eta + \eta t) + 2(\eta t - \frac{1}{2}t^2)\right]\int_0^1 M(s)ds \\ &\leq 2\eta t\int_0^1 M(s)ds \leq 3(2\eta t - t^2)\|M\|_1. \end{aligned}$$

In fact, by $1/2 < \eta \leq t \leq 1$, we have

$$3(2\eta t - t^2) - 2\eta t = 4\eta t - t^2 \geq 4\eta^2 - t^2 \geq 0.$$

For $t \in [0, \eta]$, we can have

$$\begin{aligned} \overline{w}(t) &= \int_0^1 G(t,s)M(s)ds \\ &= \int_0^t \frac{1}{2}s^2 M(s)ds + \int_t^\eta (ts - \frac{1}{2}t^2)M(s)ds + \int_\eta^1 (\eta t - \frac{1}{2}t^2)M(s)ds \\ &\leq \int_0^t \frac{1}{2}s^2 M(s)ds + \int_0^\eta (ts - \frac{1}{2}t^2)M(s)ds + \int_0^1 (\eta t - \frac{1}{2}t^2)M(s)ds \\ &\leq \left[\frac{1}{2}t^2 + \eta t + (\eta t - \frac{1}{2}t^2)\right]\int_0^1 M(s)ds \\ &= 2\eta t\int_0^1 M(s)ds \leq 2\eta t\|M\|_1. \end{aligned}$$

Then, we choose constant $C = 3 > 1$, by the above, we have

$$\overline{w}(t) \leq C\|M\|_1 q(t), \ 0 \leq t \leq 1.$$

This completes the proof of Lemma 2.4.

In the rest of the paper , we also make the following assumptions:

(H) $f\in C([0,1]\times[0,+\infty),[-\infty,+\infty))$, and there exists function $M(t)\in L^1(0,1)$, $M(t)\ge 0$ and $\int_0^1 J(s)M(s)ds<\infty$ such that

$$f(t,u)\ge -M(t),\ \forall t\in(0,1),u\ge 0.$$

Where $C\|M\|_1<1$, here C is the same constant as in Lemma 2.4 and $J(s)$ is defined in Lemma 2.1.

By Lemma 2.3, for $\theta\in(0,1/2)$, we denote a cone K of E:

$$K=\{u\in E: u(t)\ge\|u\|q(t),\theta\le t\le 1-\theta\},$$

For convenience, we set

$$\theta^*=\frac{2}{\sigma^2\int_\theta^{1-\theta}J(s)ds},\quad \theta_*=\frac{1}{\int_0^1(J(s)+M(s))ds},\quad \sigma=\min_{\theta\le t\le 1-\theta}q(t)$$

$$f_0=\lim_{u\to 0}\max_{0\le t\le 1}\frac{f(t,u)}{u},\qquad f_\infty=\lim_{u\to\infty}\min_{0\le t\le 1}\frac{f(t,u)}{u}.$$

3 The Existence of Single Positive Solution

In this section, we present our main results.

Theorem 3.1. Suppose that condition (H) hold.. Assume that for $C\|M\|_1<r<2c\|M\|_1<R$, f also satisfies

(A_1): $f(t,u)\ge MR,$ for $\sigma R/2\le u\le R,$ $0\le t\le 1;$

(A_2): $f(t,u)\le mr,$ for $0\le u\le r,$ $0\le t\le 1,$

where $M\in[\theta^*,\infty)$, $m\in(0,\theta_*],mr\ge 1.$ Then , the semipositone boundary value problem SBVP (1.1) has a solution $u\in K$ such that $\|u\|$ lies between r and R.

Theorem 3.2. Suppose that condition (H) hold. Assume that f also satisfy

(A_3): $f_0=\varphi\in[0,\theta_*-\alpha);$

(A_4): $f_\infty=\lambda\in\left(\frac{4\theta^*}{\sigma},\infty\right).$

Then, the semipositone boundary value problem SBVP (1.1) has a solution $u\in K$ such that $\|u\|$ lies between r and R.

The proof of Theorem 3.1. By Lemma 2.4, we set $w(t)=\overline{w}(t)$, Then $u(t)$ is the positive solutions of the SBVP (1.1) if and only if $\tilde{u}(t)=u(t)+w(t)$ is the positive solutions of the BVP

$$\begin{cases}u'''(t)-F(t,u(t)-w(t))=0, & t\in(0,1);\\ u(0)=u'(\eta)=u''(1)=0,\end{cases}\tag{3.1}$$

and $\tilde{u}(t)\ge w(t),$ $t\in J$, where for $t\in I$,

$$F(t,u)=H(t,u)+M(t),\quad H(t,u)=\begin{cases}f(t,u), & u\ge 0,\\ f(t,0), & u<0,\end{cases}$$

Obviously, BVP (3.1) is equivalent to the equation

$$u(t)=\int_0^1 G(t,s)F(s,u(s)-w(s))ds.\tag{3.2}$$

and consequently, it’s solution is equivalent to the fixed point problem $u=Tu$ with operator $T:E\to E$ given by

$$(Tu)(t)=\int_0^1 G(t,s)F(s,u(s)-w(s))ds.\tag{3.3}$$

Then we shall divide the rather long proof into three steps.

(I) $T:K\to K$ is completely continuous.

(a) Firstly, we proof that $T(K)\subset K$. By Lemma 2.1, (3.4), for any $u(t)\in K,\ \ t\in J,$ we have

$$(Tu)(t)=\int_0^1 G(t,s)F(s,u(s)-w(s))ds\le\int_0^1 J(s)F(s,u(s)-w(s))ds.$$

So

$$\|Tu\|\le\int_0^1 J(s)F(s,u(s)-w(s))ds. \qquad (3.4)$$

Then, by Lemma 2.1 and (3.4), for $u\in K,$ we have

$$(Tu)(t)=\int_0^1 G(t,s)F(s,u(s)-w(s))ds\ge q(t)\int_0^1 J(s)F(s,u(s)-w(s))ds\ge\|Tu\|q(t).$$

Then $T(K)\subset K$.

(b) Secondly ,we will show that T is compact operator. Let $D\subset K$ be any bounded set, then there exists a constant $M>0$ such that $\|u\|\le M,u\in D.$ Then, we have

$$\|(Tu)(t)\|\le\int_0^1 J(s)(L+M(s))ds.$$

where $L=\sup_{0\le t\le 1,\|u\|\le M} H(t,u).$ Therefore, $T(D)$ is uniformly bounded.

Next, we will show $|(T(u)'|\in L^1(0,1),u\in D.$ In fact, by (3.3), we know that if $t\in[\eta,1],$ we can get

$$\begin{aligned}|T(u)'(t)|&=\left|\int_\eta^t(\eta-s)F(s,u(s)-w(s))ds+\int_t^1(\eta-t)F(s,u(s)-w(s))ds\right|\\
&\le\left|\int_\eta^t(\eta-s)(L+M(s))ds+\int_t^1(\eta-t)(L+M(s))ds\right|\\
&\le(L+1)(\int_\eta^t(s-\eta)(1+M(s))ds+\int_t^1(t-\eta)(1+M(s))ds)\\
&\equiv\ :(L+1)h(t),\end{aligned}$$

where $h(t)=\int_\eta^t(s-\eta)(1+M(s))ds+\int_t^1(t-\eta)(1+M(s))ds$.

Then, we have

$$\begin{aligned}\int_0^1|h(t)|dt&=\int_0^1\left|\int_\eta^t(s-\eta)(1+M(s))ds+\int_t^1(t-\eta)(1+M(s))ds\right|dt\\
&=\int_\eta^1(s-\eta)(1+M(s))ds\int_s^1 dt+\int_0^1(1+M(s))ds\int_0^s(t-\eta)dt\\
&\le\int_0^1(s-\eta)(1+M(s))ds\int_s^1 dt+\int_0^1(1+M(s))(\frac{1}{2}s^2-\eta s)ds\\
&\le\int_0^1(1+M(s))ds<\infty,\end{aligned}$$

Then, $0\le\int_0^1|(T(u)'(t)|dt<\infty.$

Similar to the above, for $t\in[0,\eta],$ we can also get $0\le\int_0^1|T(u)'(t)|dt<\infty.$ Then, for any $0\le t_1\le t_2\le 1,\ u\in D,$ we have

$$|(Tu)(t_1)-(Tu)(t_2)|=\left|\int_{t_1}^{t_2}(T(u)'(t)dt\right|\le\int_{t_1}^{t_2}|T(u)'(t)|dt.$$

So by the absolute continuity of the integral, we know that $T(D)$ is equicontinuous on $[0,1]$. Thus, according to Ascoli-Arzela's theorem, we know that $T(D)$ is a relatively compact set, i.e. , T is compact operator.

Furthermore, by using Lebesgue dominated convergence theorem, we can easily obtain that $T:K\to K$ is a completely continuous operator.

(II) Next, we will discuss the positive solution of the BVP (3.1).

We define two open subset Ω_1 and Ω_2 of E:

$$\Omega_1 = \left\{u \in K : \|u\| < R\right\}, \qquad \Omega_2 = \left\{u \in K : \|u\| \le r\right\}.$$

Then, for $t \in \left[0,1\right]$ and $u \in \partial\Omega_1$, we have $u(t) - w(t) \le u(t) \le \|u\| = R$ and

$$u(t) - w(t) = u(t) - \overline{w}(t) \ge u(t) - C\|M\|_1 q(t) \ge u(t) - \frac{C\|M\|_1}{R} u(t)$$

$$= (1 - \frac{C\|M\|_1}{R})u(t) \ge \frac{1}{2}u(t),$$

so,

$$u(t) - w(t) \ge \frac{1}{2}u(t) \ge \frac{\|u\|}{2}q(t) \ge \frac{\sigma R}{2}, \quad \theta \le t \le 1-\theta.$$

And then by (A_1), we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)F(s,u(s)-w(s))ds \\ &\ge \int_0^1 J(s)q(t)(f(s,u(s)-w(s))+M(s)ds \\ &\ge \sigma\int_\theta^{1-\theta} J(s)(f(s,u(s)-w(s))ds \\ &\ge \frac{1}{2}\sigma^2 MR\int_\theta^{1-\theta} J(s)ds \ge R = \|u\|.\end{aligned}$$

Therefore, we have

$$\|Tu\| \ge \|u\|, \quad \forall u \in \partial\Omega_1.$$

Then by Theorem1.1, we have

$$i\left(T,\Omega_1,K\right) = 0. \qquad (3.5)$$

On the other hand, for $\forall u \in \partial\Omega_2$, we have $u(t) - w(t) \le u(t) \le \|u\| = r$ and

$$u(t) - w(t) = u(t) - \overline{w}(t) \ge u(t) - C\|M\|_1 q(t) \ge u(t) - \frac{C\|M\|_1}{r} u(t)$$

$$= (1 - \frac{C\|M\|_1}{r})u(t) \ge 0,$$

and then by (A_2), we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)F(s,u(s)-w(s))ds \\ &\le \int_0^1 J(s)(f(s,u(s)-w(s))+M(s)ds \\ &\le \int_0^1 (J(s)mr + M(s))ds \\ &\le mr\int_0^1 (J(s)+M(s))ds \le r = \|u\|.\end{aligned}$$

Therefore, we have

$$\|Tu\| \le \|u\|, \quad \forall u \in \partial\Omega_2$$

Then by Theorem 1.1, we have

$$i\left(T,\Omega_2,K\right) = 1. \qquad (3.6)$$

Therefore, by (3.5), (3.6), $r < R$, we have

$$i\left(T,\Omega_1 \setminus \overline{\Omega}_2, K\right) = -1.$$

Then operator T has a fixed point $\tilde{u} \in \left(\Omega_1 \setminus \overline{\Omega}_2\right)$, and $r \leq \left\|\tilde{u}\right\| \leq R$.

(III) Finally, we will show that $\tilde{u}(t) \geq w(t),\ \ t \in \left(\theta, 1-\theta\right)$.

By, Lemma 2.3 and 2.4, for $t \in \left(\theta, 1-\theta\right)$, we have

$$\tilde{u}(t) \geq \left\|\tilde{u}\right\| q(t) \geq r q(t) > C\left\|M\right\|_1 q(t) \geq \overline{w}(t) = w(t),$$

i.e., $u(t) = \tilde{u}(t) - w(t)$ is the positive solution of SBVP (1.1). This completes the proof of Theorem 3.1.

The proof of Theorem 3.2. The proof of Theorem 3.2 is almost the same as that of Theorem 3.1. All we need to do is to discuss the operator T which is defined by (3.3).

First, by $f_0 = \varphi \in \left[0, \theta_* - \alpha\right)$, for $\varepsilon = \theta_* - \alpha - \varphi$, there exists a positive number $\rho > C\left\|M\right\|_1$, $\rho(\theta_* - \alpha) \geq 1$, as $0 \leq u \leq \rho$, $\mathrm{u} \neq 0$, we have

$$f(t,u) \leq (\varphi + \varepsilon)u = (\theta_* - \alpha)\rho. \qquad (3.7)$$

Then let $r = \rho,\ \ m = \theta_* - \alpha \in (0, \theta_*)$, thus by (3.7), we have

$$f(t,u) \leq mr, \qquad 0 \leq u \leq r.$$

So condition (A_2) holds.

Next, by condition (A_4), $f_\infty = \lambda \in \left(\frac{4\theta^*}{\sigma}, \infty\right)$, then for $\varepsilon = \lambda - \frac{4\theta^*}{\sigma}$, there exists an appropriately big positive number $R \neq r$, as $u \geq \sigma R / 2$, we have

$$f(t,u) \geq (\lambda - \varepsilon)u \geq \left(\frac{4\theta^*}{\sigma}\right) . \frac{\sigma R}{2} = 2\theta^* R\ , \qquad (3.8)$$

Let $M = 2\theta^* > \theta^*$, thus by (3.8), condition (A_1) holds. Therefore by Theorem 3.1 we know that the results of Theorem 3.2 holds. The proof of Theorem 3.2 is complete.

4 Application

Example 4.1. Consider the following boundary value problem (BVP)

$$\begin{cases} u'''(t) - \left[u(t)\ln(1+u(t)) - \frac{1}{3}t^2\right] = 0, & 0<t<1, \\ u(0) = u'(2/3) = u''(1) = 0. \end{cases} \qquad (5.1)$$

We can easily show that $f(t,u) = u(t)\ln(1+u(t)) - \frac{1}{3}t^2$ satisfy:

$$f(t,u) = u(t)\ln(1+u(t)) - \frac{1}{3}t^2 \geq -\frac{1}{3}t^2 = -M(t),$$

Obviously, $\int_0^1 M(s)ds = \int_0^1 \frac{1}{3}t^2 dt = \frac{1}{9} < 1$. So, condition (H) holds.

Next, we can easily know that $f_0 = 0,\ f_\infty = \infty$. So condition $(A_3), (A_4)$ hold.

Therefore, we can choose $C = 3$ (see Lemma 2.4) and r, R such that $C\left\|M_1\right\| = 1/3 < r \leq 2C\left\|M_1\right\| = 2/3 < R$. Then, by Theorem 3.2, we can show that for the SBVP (4.1) have at least two positive solutions $u(t)$ and $r \leq \left\|u\right\| \leq R$.

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